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Finite groups with a certain number of values of the Chermak–Delgado measure

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In this paper, we study the finite groups whose Chermak–Delgado measure has exactly k values. We will focus especially on the case $k = 2$. These groups determine an interesting class of p-groups containing cyclic groups of prime order and extraspecial p-groups.

Keywords: Chermak–Delgado measure; Chermak–Delgado lattice; subgroup lattice; generalized quaternion 2-group; extraspecial p-group; outer abelian p-group; p-group of maximal class.

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1. Introduction

Throughout this paper, let G be a finite group and $L(G)$ be the subgroup lattice of G. Denote by

$$m_G(H) = |H||C_G(H)|$$

the *Chermak–Delgado measure* of a subgroup H of G and let

$$m^*(G) = \max\{m_G(H) \mid H \leq G\} \quad \text{and} \quad \mathcal{CD}(G) = \{H \leq G \mid m_G(H) = m^*(G)\}.$$

Then the set $\mathcal{CD}(G)$ forms a modular, self-dual sublattice of $L(G)$, which is called the *Chermak–Delgado lattice* of G. It was first introduced by Chermak and Delgado [7], and revisited by Isaacs [9]. In the last years, there has been a growing interest in understanding this lattice (see e.g. [3–6, 8, 10–13, 15, 18, 20]). We recall several important properties of the Chermak–Delgado measure that will be used in our paper:

- if $H \leq G$, then $m_G(H) \leq m_G(C_G(H))$, and if the measures are equal then $C_G(C_G(H)) = H$;

- if $H \in \mathcal{CD}(G)$, then $C_G(H) \in \mathcal{CD}(G)$ and $C_G(C_G(H)) = H$;
- the minimum subgroup $M(G)$ of $\mathcal{CD}(G)$ (called the *Chermak–Delgado subgroup* of G) is characteristic, abelian, and contains $Z(G)$.

We remark that the Chermak–Delgado measure associated to a finite group G can be seen as a function

$$m_G : L(G) \to \mathbb{N}^*, \quad H \mapsto m_G(H), \quad \forall\, H \in L(G).$$

The starting point for our discussion is given by [16, Corollary 3], which states that there is no finite nontrivial group G such that $\mathcal{CD}(G) = L(G)$. In other words, m_G has at least two distinct values for every finite nontrivial group G. This leads to the following natural question:

Given an integer $k \geq 2$, which are the finite groups G whose Chermak–Delgado measure m_G has exactly k values?

The paper is organized as follows. In Sec. 2, we present some general results on the above groups. Section 3 deals with the case $k = 2$. In the final section, some further research directions are indicated.

We recall several basic definitions:

— a *generalized quaternion* 2*-group* is a group of order 2^n, $n \geq 3$, defined by the presentation

$$Q_{2^n} = \langle a,\, b \mid a^{2^{n-1}} = 1,\, a^{2^{n-2}} = b^2,\, b^{-1}ab = a^{-1} \rangle;$$

— a finite p-group G is said to be *extraspecial* if $Z(G) = G' = \Phi(G)$ has order p;

— a finite p-group G is said to be *outer abelian* if G is non-abelian, but every proper quotient group of G is abelian;

— a finite p-group G of order p^n is said to be *of maximal class* if the nilpotence class of G is $n - 1$.

The following well-known facts on p-groups will be useful to us. The first two appear in [14, Eqs. (4.26) and (4.4)], the third in [19, Corollary 10], while the fourth in [1, Proposition 1.8].

- Any group of order p^4 contains an abelian subgroup of order p^3.
- A finite p-group G has a unique subgroup of order p if and only if either it is cyclic or $p = 2$ and $G \cong Q_{2^n}$ for some $n \geq 3$.
- A finite p-group G is outer abelian if and only if $|G'| = p$ and $Z(G)$ is cyclic, and G is one of the following non isomorphic groups:
  (1) $M(n,1) = \langle a, b \mid a^{p^n} = b^p = 1, a^b = a^{1+p^{n-1}} \rangle$, $n \geq 3$;
  (2) an extraspecial p-group;
  (3) $G = E * A$, where E is an extraspecial p-group and $A \cong M(n,1)$, $n \geq 3$;
  (4) $G = E * A$, where E is an extraspecial p-group and $A \cong C_{p^t}$, $t \geq 2$.
- A finite p-group G is of maximal class if and only if it has a subgroup A of order p^2 such that $C_G(A) = A$.

2. Some General Results

First of all, we indicate a lower bound for the number of values of the Chermak–Delgado measure associated to a finite group.

Theorem 2.1. *Let G be a finite group. For each prime p dividing the order of G and $P \in \mathrm{Syl}_p(G)$, let $|Z(P)| = p^{n_p}$. Then*

$$|\mathrm{Im}(m_G)| \geq 1 + \sum_p n_p. \tag{1}$$

Proof. Let p be a prime dividing the order of G and $P \in \mathrm{Syl}_p(G)$ with $|P| = p^{m_p}$ and $|Z(P)| = p^{n_p}$. For each such p and each i, $1 \leq i \leq n_p$, there is a subgroup $H_{p,i} \leq Z(P)$ with $|H_{p,i}| = p^i$. Since $P \subseteq C_G(H_{p,i})$, it follows that the exponent of p in $m_G(H_{p,i})$ is $i + m_p$, and so $m_G(H_{p,i}) \neq m_G(H_{p,j})$ for $i \neq j$. We also observe that we have $m_G(H_{p_1,i}) \neq m_G(H_{p_2,j})$ for $p_1 \neq p_2$. Then $\mathrm{Im}(m_G)$ has at least $1 + \sum_p n_p$ distinct elements, namely $m_G(1)$ and $m_G(H_{p,i})$, where p runs over all prime divisors of $|G|$ and $i = 1, 2, \ldots, n_p$. This completes the proof. □

Note that for a p-group G we have equality in (1) if and only if $G \in \mathcal{CD}(G)$. This happens for large classes of p-groups, such as for all abelian p-groups. Assume now that $|G| = p_1^{m_{p_1}} p_2^{m_{p_2}} \cdots p_k^{m_{p_k}}$ with $k \geq 2$ and that equality occurs in Eq. (1). Then either $m_G(G) = m_G(1)$ or there are $i \in \{1, \ldots, k\}$ and $1 \leq j \leq n_{p_i}$ such that $m_G(G) = m_G(H_{p_i,j})$. These conditions easily lead to $Z(G) = 1$ or $Z(G) = H_{p_i,j}$. We observe that none of them assure the equality in Eq. (1), as show the examples $G = S_4$ and $G = D_{12}$, respectively. We are also able to give several examples of non-p-groups where equality occurs in Eq. (1), such as A_4 and all non-abelian groups of order pq with p, q distinct primes.

The following theorem gives a precise formula of computing $|\mathrm{Im}(m_G)|$ for finite abelian groups G.

Theorem 2.2. *Let G be a finite abelian group of order n. Then*

$$|\mathrm{Im}(m_G)| = \tau(n),$$

where $\tau(n)$ denotes the number of divisors of n.

Proof. Let H be a subgroup of order d of G. Then

$$m_G(H) = |H||C_G(H)| = |H||G| = dn$$

and $d \,|\, n$. Conversely, let d be a divisor of n. Since G is abelian, we infer that there is a subgroup $H \leq G$ such that $|H| = d$, and so $m_G(H) = dn$. Thus,

$$\mathrm{Im}(m_G) = \{dn : d \,|\, n\},$$

which implies that

$$|\mathrm{Im}(m_G)| = \tau(n),$$

as desired. □

We remark that Theorem 2.2 can be used to determine all finite abelian groups G whose Chermak–Delgado measure has a small number of values.

Corollary 2.3. *Let G be a finite abelian group with $|\mathrm{Im}(m_G)| = k$.*

(a) *If $k = 2$, then $G \cong C_p$ for some prime p.*
(b) *If $k = 3$, then either $G \cong C_{p^2}$ or $G \cong C_p \times C_p$ for some prime p.*

3. The Case $k = 2$

In this section, we study the class $\mathcal{C}$ of finite groups G whose Chermak–Delgado measure m_G has exactly two values. Note that the abelian groups in $\mathcal{C}$ have been determined in the above corollary. We easily infer that $\mathcal{C}$ is not closed under subgroups, homomorphic images, direct products or extensions. Also, by Theorem 2.1 one obtains that.

Proposition 3.1. *All groups in $\mathcal{C}$ are p-groups with center of order p.*

Since our study can be reduced to p-groups and it is completely finished for abelian groups, in what follows, we will suppose that G is a non-abelian p-group of order p^n $(n \geq 3)$ belonging to $\mathcal{C}$. Then:

(a) $G \in \mathcal{CD}(G)$;
(b) $\mathrm{Im}(m_G) = \{p^n, p^{n+1}\}$, and consequently $m^*(G) = p^{n+1}$;
(c) $Z(G)$ is the unique minimal normal subgroup of G, and consequently $Z(G) \subseteq G' \subseteq \Phi(G)$;
(d) $HZ(G) \in \mathcal{CD}(G)$, $\forall\, H \leq G$ satisfying $Z(G) \nsubseteq H$.

If $Z(G) \nleq H$, then $H \notin \mathcal{CD}(G)$. Since $C_G(H) = C_G(HZ(G))$, it follows that $m_G(H) \neq m_G(HZ(G))$, and consequently $HZ(G) \in \mathcal{CD}(G)$.

There are many examples of finite non-abelian p-groups G such that $\mathcal{CD}(G) = \{Z(G), G\}$ (see e.g. [5, Corollary 2.2 and Proposition 2.3]). Using Corollary 2.3(a), and the above item (d), we are able to prove that the intersection between this class of groups and $\mathcal{C}$ is empty.

Corollary 3.2. *$\mathcal{C}$ does not contain non-abelian p-groups G with $\mathcal{CD}(G) = \{Z(G), G\}$.*

Proof. Assume that $\mathcal{C}$ contains a non-abelian p-group G satisfying $\mathcal{CD}(G) = \{Z(G), G\}$.

If G possesses a minimal subgroup $H \neq Z(G)$, then $HZ(G) \in \mathcal{CD}(G)$ by (d). On the other hand, we obviously have $HZ(G) \neq Z(G)$, and since $\mathcal{CD}(G) = \{Z(G), G\}$, we get $HZ(G) = G$. Then $|G| = p^2$, implying that G is abelian, a contradiction.

If $Z(G)$ is the unique subgroup of order p in G, then G is a generalized quaternion 2-group, i.e. $p = 2$ and

$$G \cong Q_{2^n} = \langle a, b \,|\, a^{2^{n-1}} = 1,\ a^{2^{n-2}} = b^2,\ b^{-1}ab = a^{-1} \rangle \quad \text{for some } n \geq 3.$$

It results that G has a cyclic maximal subgroup $H \cong \langle a \rangle$. So,

$$m_G(H) = 2^{2n-2} \leq 2^{n+1} = m^*(G),$$

which means $n \leq 3$. Since G is non-abelian, we get $n = 3$, that is $G \cong Q_8$. Then $\mathcal{CD}(G)$ is a quasi-antichain of width 3, contradicting the hypothesis. □

Next, we will focus on giving examples of non-abelian p-groups in $\mathcal{C}$.

Theorem 3.3. *All extraspecial p-groups are contained in $\mathcal{C}$.*

Proof. Let G be an extraspecial p-group. It is well known that $\mathcal{CD}(G)$ consists of all subgroups H of G containing $Z(G)$ (see e.g. [8, Example 2.8] or [17, Theorem 4.3.4]). Consequently, all these subgroups have the same Chermak–Delgado measure. On the other hand, by [2, Lemma 2.6] any subgroup H of G with $Z(G) \nsubseteq H$ satisfies $m_G(H) = |G|$. Thus the function m_G has exactly two values, as desired. □

Using GAP, we are also able to give an example of a nonextraspecial non-abelian p-group in $\mathcal{C}$, namely SmallGroup(32,8):

$$G = \langle a,\, b,\, c \,|\, a^4 = 1,\, b^4 = a^2,\, c^2 = bab^{-1} = a^{-1},\, ac = ca,\, cbc^{-1} = a^{-1}b^3 \rangle.$$

Note that the nilpotence class of G is 3. Also, $\mathcal{CD}(G)$ is described in [17, Lemma 4.5.16 and Corollaries 4.5.20 and 4.5.21].

We observe that all non-abelian groups of order p^3 belong to $\mathcal{C}$ because they are extraspecial. The same thing cannot be said about non-abelian groups of order p^4: such a group G has an abelian subgroup A of order p^3, and so

$$m^*(G) \geq m_G(A) = p^6 > p^5,$$

implying that G is not contained in $\mathcal{C}$. This argument can be extended in the following way.

Proposition 3.4. *If a non-abelian group of order p^n contains an abelian subgroup of order $\geq p^{[\frac{n+3}{2}]}$, then it does not belong to $\mathcal{C}$.*

From Proposition 3.4, we easily infer that certain groups does not belong to $\mathcal{C}$. For example, no group of order 64 is contained in $\mathcal{C}$ because such a group possesses an abelian subgroup of order 16. Another interesting application of Proposition 3.4 is the following.

Theorem 3.5. *Let G be a finite p-group of nilpotence class 2 contained in $\mathcal{C}$. Then G is extraspecial.*

Proof. Since the nilpotence class of G is 2, we have that $G/Z(G)$ is abelian and so $G' \subseteq Z(G)$. By Proposition 3.1, we get $G' = Z(G)$, which implies that G is an outer abelian p-group. Then G belongs to one of the four classes of groups (1)–(4) described in Sec. 1.

We observe that $M(n,1)$ has a cyclic subgroup of order p^n, namely $\langle a\rangle$, and $n \geq [\frac{n+4}{2}]$ for $n \geq 3$. Thus, it cannot be contained in $\mathcal{C}$ by Proposition 3.4. Also, it is easy to see that a central product $E * A$, where E is an extraspecial p-group of order p^{2m+1} and $A \cong M(n,1)$, $n \geq 3$, always has an abelian subgroup of order p^{m+n}. Since $m+n \geq [\frac{2m+n+4}{2}]$ for $n \geq 3$, by Proposition 3.4, we infer that $E * A$ does not belong to $\mathcal{C}$. Similarly, a central product $E*A$, where E is an extraspecial p-group of order p^{2m+1} and $A \cong C_{p^t}$ with $t \geq 2$, always has an abelian subgroup of order p^{m+t}. If $t \geq 3$ then $m+t \geq [\frac{2m+t+4}{2}]$, implying that $E * A$ is not contained in $\mathcal{C}$. If $t = 2$, it suffices to observe that the center of $E * A$ is of order p^2, and consequently $E * A$ is not contained in $\mathcal{C}$ by Proposition 3.1. These shows that the unique possibility is that G be an extraspecial p-group, as desired. □

Our last result shows that the non-abelian groups of order p^3 are in fact the unique p-groups of maximal class in $\mathcal{C}$.

Theorem 3.6. *Let G be a finite p-group of maximal class contained in $\mathcal{C}$. Then G is non-abelian of order p^3.*

Proof. Obviously, G is non-abelian. Let $|G| = p^n$. We know that G possesses a subgroup A of order p^2 such that $C_G(A) = A$. It follows that $m_G(A) = p^4$, and therefore, we have either $n = 3$ or $n = 4$. Since the case $n = 4$ is impossible, we get $n = 3$, as desired. □

4. Further Research

We end our paper by indicating three natural open problems concerning the above results.

Problem 1. Which are the pairs (p, n), where p is a prime and n is a positive integer, such that $\mathcal{C}$ contains groups of order p^n?

Note that all pairs (p, n) with n odd satisfy this property by Corollary 2.3(a), and Theorem 3.3.

Problem 2. Study the class $\mathcal{C}'$ of finite groups G whose Chermak–Delgado measure m_G has exactly three values.

We know that $\mathcal{C}'$ contains all abelian p-groups of order p^2 by Corollary 2.3(b); it also contains several classes of non-abelian groups, such as the non-abelian groups of order pq (p, q primes), and in particular the dihedral groups D_{2p} with p an odd prime.

Problem 3. Given a finite group G, we consider the natural action of $\mathrm{Aut}(G)$ on $L(G)$ and denote by k the number of distinct orbits. Since every two subgroups in the same orbit have the same Chermak–Delgado measure, we have

$$|\mathrm{Im}(m_G)| \leq k. \tag{2}$$

Determine the finite groups G for which equality occurs in (2).

Clearly, cyclic groups satisfy this property. Note that there are also examples of noncyclic groups satisfying it, such as elementary abelian p-groups.

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